Frobenius Algebras & 2D TQFTs

The category of 2-dimensional cobordisms 2Cob, Generators and relations of 2Cob

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1 Hors d'oeuvre

The category ***2Cob*** is the category...

- whose objects are the closed oriented 1-manifolds.
- whose arrows are the diffeomorphism classes of oriented cobordisms between them.

It can be described explicitly. This is not possible in general (say, for ***nCob***), since the complete classification of surfaces is a privilege currently reserved for dimension 2 and lower. The category of 2-dimensional cobordisms thus hits the sweet spot between not trivial like ***1Cob*** while still being solvable. While some (but definitely not all) results presented here applicable to 2-dimensional cobordisms can be generalised to higher dimensions, we will always restrict ourselves to 2 dimensions, even if not explicitly stated.

Furthermore, ***2Cob*** bears significance in certain discussions about theoretical models in physics [1]! Topological Quantum Field Theories (TQFTs) possesses certain features one expects from a theory of quantum gravity. It serves as a toy model and reference frame in which one can do calculations and gain experience before embarking on the quest for the full-fledged theory, which is expected to be much more complicated.

Roughly, the closed manifolds represent space, while the cobordisms represent space-time. In the third introductory lecture, we have encountered a functor from ***2Cob*** to $\boldsymbol{Vect}_k$ bearing a monoidal structure, aptly called a (2-dimensional) TQFT. The fact that the disjoint union goes to tensor product expresses the common principle in quantum mechanics that the state space of two independent systems is the tensor product of the two state spaces. The associated vector spaces are then the state spaces, and an operator associated to a space-time is the time-evolution operator[1].

The topological part in the name comes from the observation that time-evolution operators do not depend on any additional structure on space-time (like Riemannian metric or curvature), only on the topology.

[1] also called transition amplitude or Feynman path integral.

2 Skeleton categories

Since ***2Cob*** is a large category (i.e. its objects are, in a sense, too numerous to form a set), we cannot get hold of something we will soon know as a generating set. It is therefore a wise idea to try to reduce ***2Cob*** to a more manageable size. Since we already discussed that any closed, orientable 1-manifold is diffeomorphic to a disjoint union of circles, we arrive at a mathematically dense but still intuitive idea.

Recall that

a) Any connected, closed and oriented 1-manifold is diffeomorphic to the circle S^1 (since in this case they are diffeomorphic if and only if they are homotopic).

b) Any closed and oriented 1-manifold with n components is diffeomorphic to the disjoint union of n copies of S^1.

This motivates us to define an equivalence relation between closed, orientable manifolds:

$$M \sim N \quad \overset{\text{def.}}{\Longleftrightarrow} \quad M \cong N$$

The collection of equivalent manifolds is called an isomorphism class.

Definition 2.1. A **skeleton** $\mathcal{Z}$ of a category $\mathcal{C}$ is the minimal subcategory of $\mathcal{C}$ that is still equivalent to $\mathcal{C}$ as categories.

More technically, it is a full subcategory comprising exactly one object from every isomorphism class.

While the second characterisation of $\mathcal{Z}$ is more instructive, the minimality condition of the first characterisation is very revealing about the nature of skeletons. They are equivalent to their "parent" category $\mathcal{C}$ since there exists an equivalence (full, faithful, and essentially surjective functor) between them, which in this case is the embedding $\mathcal{Z} \hookrightarrow \mathcal{C}$. But $\mathcal{Z}$ is minimal in the sense that, by removing any objects we lose essential surjectivity, and by removing any arrows we lose fullness. *A skeleton $\mathcal{Z} \subset \mathcal{C}$ captures the essential structure of $\mathcal{C}$.*

Example 2.2. *The category $\boldsymbol{FinSet}_0$ is the category with finite sets as objects and invertible maps/isomorphisms between them as arrows. Note however that for two sets S_1, S_2 to be isomorphic, they need to have the same cardinality, $|S_1| = |S_2|$. We can already see that some grouping emerges, as there are no two objects with different cardinalities in this set that are connected by an arrow.*

The skeleton of $\boldsymbol{FinSet}_0$ is defined as $\mathcal{Z} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$, i.e. choosing one representative set per cardinality as objects. Now, there are arrows from $\boldsymbol{n}$ to $\boldsymbol{m}$ if and only if $\boldsymbol{n} = \boldsymbol{m}$. The arrows from $\boldsymbol{n}$ to $\boldsymbol{n}$ are precisely the permutations of the elements in the set $\boldsymbol{n} = \{0, 1, \dots, n-1\}$, which themselves form a 1-object category. Thus we have $\mathcal{Z} = \coprod_{n \in \mathbb{N}_0} \sigma_n$ as categories (see talk 9 for further details).

Note that there is no canonical way of choosing representatives of isomorphism classes, however any two skeletons are isomorphic.

3 Group-theoretic intuition

The explicit description of the skeleton of ***2Cob*** , which we yet have to get to know, is done in terms of generators and relations, something the aspiring mathematician likely first encountered in group theory.

Definition 3.1. The **presentation** of a group G is written as $\langle S|R\rangle$ and consists of a set of generators S such that any element of G can be written as a composition of powers of elements (and their inverses) in S, and a set R of relations among those generators. R is a set of equalities involving compositions of powers of generators to conclusively ascertain G. Formally, G is isomorphic to the quotient of a free group on S by the normal subgroup generated by R.

Example 3.2. *Presentations of groups*

- *The free group generated by a single element is denoted F_1 and is presented by $\langle a|\varnothing\rangle$. Therefore, $F_1 = \{a^n|n \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}$ with addition as group operation.*
- *The cyclic group of order n is denoted C_n and is presented by $\langle a|a^n = e\rangle$. Therefore, $C_n = \{a^0 = e, a^1, \ldots, a^{n-1}\}$, which can be interpreted as the set of rotational symmetries for the n-sided polygon.*
- *The dihedral group of order $2n$ is denoted D_n and is presented by $\langle r, f|r^n = f^2 = (rf)^2 = e\rangle$. Therefore, D_n can be interpreted as the set of symmetries for the n-sided polygon, where we allow rotations r as well as flips f.*

These are some nice simple examples to get back into group theory. Consider now the next example, which will be vitally important later:

Example 3.3. *The symmetric group of order n is denoted σ_n and is the group of permutations acting on $n \geq 4$ letters/objects $\{x_1, \ldots, x_n\}$. They are generated by adjacent transpositions, $S = \{\tau_i = (x_i x_{i+1})$[2]$|i = 1, \ldots, n-1\}$, which means that any permutation can be created/reached by subsequent flips of neighbouring letters.*
Secondly, the generators are subject to the following relations:

$$\begin{aligned} \tau_i\tau_i &= e && \forall i \\ \tau_i\tau_j\tau_i &= \tau_j\tau_i\tau_j && \forall j = i+1 \\ \tau_i\tau_j &= \tau_j\tau_i && \forall j > i+1 \end{aligned}$$

The first relation means that two subsequent transposition just invert each other. The third relation means that transpositions for disjoint pairs commute. The second relation is not so simple to grasp in words, but maybe we don't have to.

Consider σ_4, and visualise the letters $\{x_1, x_2, x_3, x_4\}$ by drawing them as a column of dots. Then, we can visualise the generators and relations by expressing permutations as connecting graphs, almost as if we were tying our shoes. This leaves us with:

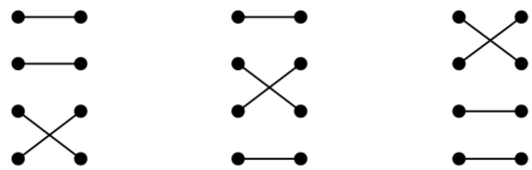


Figure 1: Visualisation of the generators of σ_4.

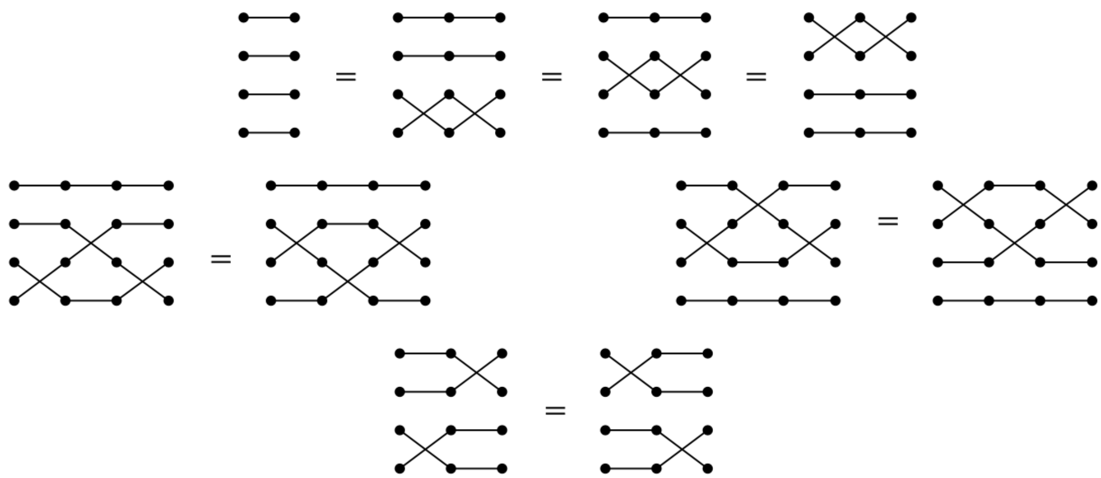


Figure 2: Visualisation of the relations of σ_4.

However, don't you think that there is some redundant visual information present? For example, do we really need to draw the parallel top/bottom line for the second relation in order to hold? Indeed, this leads us to ***Paralleling as generating concept****. Notice for example that the generators in σ_4 are just the transposition element ($\in \sigma_2$) in disjoint union (combined in parallel) with identity permutations. This is the only generator in σ_2 (it is the*

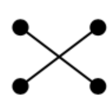


Figure 3: Visualisation of the transposition.

only non-trivial element), and if we allow paralleling as a generating concept, it also suffices to generate σ_4. With parallel coupling, we can further reduce the amount of relations to two:

[2]Cycle notation.

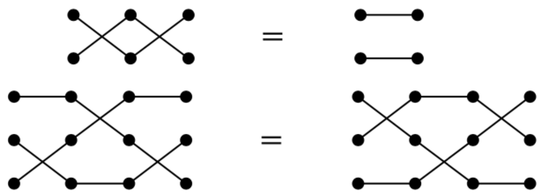


Figure 4: Visualisation of the relations modulo paralleling.

Compare now 3.1 to the following definition, which will be essential for the rest of the chapter:

Definition 3.4. The **generators** for a category $\mathcal{C}$ is a set of arrows S such that every arrow in $\mathcal{C}$ can be obtained by composing the arrows of S. *If* $\mathcal{C}$ *additionally has a monoidal structure*, S is such that every arrow in $\mathcal{C}$ can be obtained by combining composition of arrows in S and the associated monoidal functor $\square$.

A **relation** is the equality of two ways of writing a given arrow in $\mathcal{C}$. A set of relations R is **complete** if every other relation can be obtained by combining elements of R.

In talk 2 and 3 we discovered that ($\textbf{\textit{2Cob}}, \coprod, \varnothing, twist$) is a (symmetric) monoidal category, so we expect to obtain every arrow in ***2Cob*** by combining composition of arrows in S and the disjoint union $\coprod$.

4 Considerations of cobordisms

The notion of invertible cobordisms gives us a sense in which closed manifolds are diffeomorphic. Check talk 3 for the proof that the cylinder cobordism indeed is the identity arrow (modulo diffeomorphism) for any object in the category ***2Cob*** .

Definition 4.1. The cobordism $M : \Sigma_0 \to \Sigma_1$ is **invertible** if there exists a cobordism $M^{-1} : \Sigma_1 \to \Sigma_0$ such that MM^{-1} is diffeomorphic to the cylinder $\Sigma_0 \times I$ and $M^{-1}M$ is diffeomorphic to the cylinder $\Sigma_1 \times I$.

Lemma 4.2. *Let $M : \Sigma_0 \to \Sigma_1$ be invertible, and M be connected as a manifold. Then are Σ_0 and Σ_1 connected.*

Proof. By assumption $MM^{-1} \cong \Sigma_0 \times I \equiv C_0$, which is "horizontally connected". This means that every point of C_0 is connected to some point of its in-boundary. But this is also the in-boundary of M which is assumed to be connected, so C_0 is connected. And since for an arbitrary Σ, the cylinder $\Sigma \times I$ has the same number of connected components as Σ, Σ_0 (which is the base of C_0) is connected as well. The same arguments hold for $M^{-1}M \cong \Sigma_1 \times I \equiv C_1$ as well, thus Σ_1 is connected as well. $\square$

Lemma 4.3. *Let $M : \Sigma_0 \to \Sigma_1$, $M' : \Sigma_0' \to \Sigma_1'$ be two cobordisms such that the disjoint union cobordism $M \coprod M' : \Sigma_0 \coprod \Sigma_0' \to \Sigma_1 \coprod \Sigma_1'$ is invertible. Then are M, M' invertible as well.*

Corollary 4.4. *Let $M : \Sigma_0 \to \Sigma_1$ be invertible. Then Σ_0 and Σ_1 have the same number of connected components.*

Proof. The case of M having one component is covered in 4.2, so let M have more than one connected component. As an inductive step, let it be true for n connected components, that its boundaries Σ_0 and Σ_1 have n connected components as well. Then if M has $n+1$ connected components, we note that this is a disjoint union of a cobordism with n connected components and a connected cobordism. So inductively via 4.3 we arrive at the desired conclusion. □

Proposition 4.5. *Let Σ_0, Σ_1 be closed oriented 1-manifolds. They are diffeomorphic if and only if there is an invertible cobordism between them.*

Proof. ($\Longrightarrow$) From talk 3 we have seen that for a given diffeomorphism $\Sigma_0 \cong \Sigma_1$, we can use the cylinder construction, which is its own inverse and is indeed diffeomorphic to a cylinder, so we have found an invertible cobordism between them.

($\Longleftarrow$) By assumption there is an invertible cobordism. By 4.4, Σ_0 and Σ_1 are closed, oriented 1-manifolds. But then they are both diffeomorphic to a disjoint union of the same number of circles, and thus indeed they are diffeomorphic to each other. □

With the following corollary, we are finally ready to discuss the objects of the skeleton of ***2Cob*** .

Corollary 4.6. *Two objects of **2Cob** (two closed, oriented 1-manifolds) are in the same isomorphism class (that is, there exists an invertible cobordism between them) of **2Cob** if and only if they have the same number of components.*

The skeleton of ***2Cob*** is the category…

- whose objects form the set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$ where $\mathbf{0}$ denotes the empty manifold, $\mathbf{1}$ a circle S^1, and $\mathbf{n}$ the disjoint union of n copies of S^1.
- whose arrows are the diffeomorphism classes of oriented cobordisms between them.

By slight abuse of notation, we denote this skeleton ***2Cob*** as well. For everything that follows, whenever we refer to ***2Cob*** and its properties, chances are high that we are actually talking about its skeleton. Luckily they are equivalent as categories, so in our discussion here this will not be an issue.

One might be tempted to think that since we included the disjoint union as an allowed operation for generating, we can focus solely on connected cobordisms. This is however not entirely true! The twist cobordism which acts on a disjoint union of two non-empty objects, $\Sigma_0 \coprod \Sigma_1 \xrightarrow{\sim} \Sigma_1 \coprod \Sigma_0$, is not diffeomorphic to the cylinder! This is precisely because a necessary condition for two cobordisms to be equal modulo diffeomorphism is that they both treat the boundary the same way.

Remark 4.7. *For two sets A and B, generally we have $A \bigcup B = B \bigcup A$, but $A \coprod B \neq B \coprod A$ (there is however a canonical isomorphism $A \coprod B \cong B \coprod A$, realised through the twist map)! From the categorial point of view, the disjoint union is the coproduct in* ***Set*** *with canonical inclusion maps.*

The point is that even for $\Sigma \coprod \Sigma$, where Σ might be a set or a 1-manifold for example, the two copies are not identical and can be distinguished. A neat way of realising this is by labelling every element in the first set with a lower index 1 and every element in the second set with a lower index 2.

In light of 4.7, we can distinguish two copies of Σ by choosing a point in $\Sigma \coprod \Sigma$ and differentiating which copy contains the point and which does not. $twist_{\Sigma \coprod \Sigma} \ncong id_{\Sigma \coprod \Sigma}$ then because under the identity (cylinder), the point stays on the same connected component, but it does not so under the twist map.

5 Generators for *2Cob*

Theorem 5.1. *The monoidal category* ***2Cob*** *is generated under composition*[3] *and disjoint union*[4] *by the following six cobordisms:*

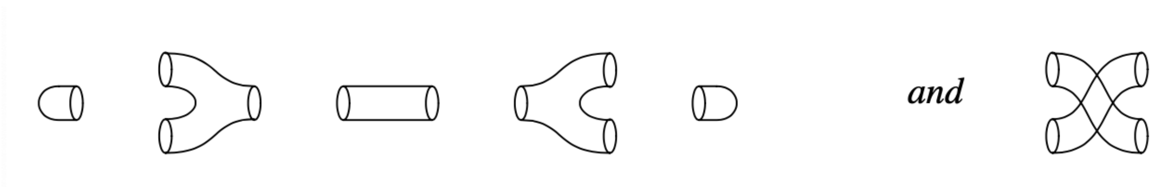


Figure 5: We shall call them, from left to right: left cap, left pants, cylinder, right pants, right cap, and twist.

Let us emphasise this again: We are really talking about the skeleton of ***2Cob*** , and not about ***2Cob*** itself, because in general we cannot obtain a (finite) generating set for a large category. We are content to describe the generators of the skeleton nonetheless, because by definition it still captures the essential structure of its parent category.

We will discuss two different proof of 5.1 in varying depth. Recall that the objects of (the skeleton of) ***2Cob*** are exactly $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}, \ldots\}$.

5.1 Proof by normal form and permutation

Definition 5.2. The **normal form** of a connected (oriented & compact) surface with n in-boundaries, m out-boundaries, and genus g is a decomposition of the surface into 3 basic cobordisms.

[3]serial connection

[4]parallel connection

- The in-part is a cobordism $\mathbf{n} \to \mathbf{1}$ consisting of $n-1$ (for $n > 0$) left pants composed one after the other on the bottom leg, together with the appropriate number of cylinders on top. If $n = 0$, it consists of a single left cap.
- The middle part is a cobordism $\mathbf{1} \to \mathbf{1}$ consisting of g pairs of right pants composed with left pants.
- The out-part is a cobordism $\mathbf{1} \to \mathbf{m}$ consisting of $m-1$ (for $m > 0$) right pants composed one after the other on the bottom leg, together with the appropriate number of cylinders on top. If $m = 0$, it consists of a single right cap.

Recall the naming convection from Figure 5. This is one of those definitions who are much easier to understand with an example, so here is one:

Example 5.3.

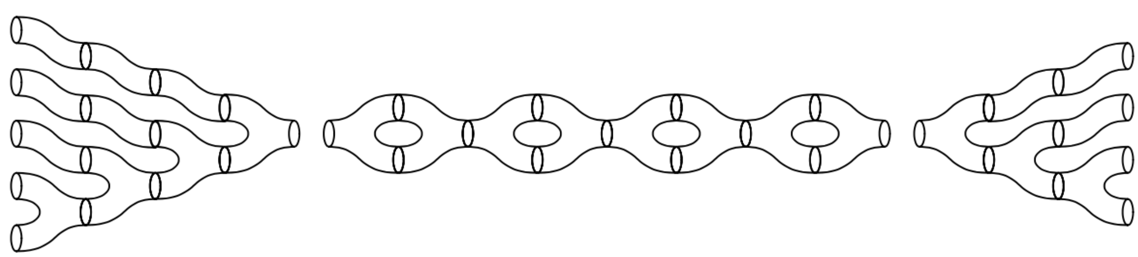


Figure 6: Normal form of a surface with $n = 5$, $g = 4$, and $m = 4$.

This construction leads to the following result:

Lemma 5.4. *Every connected 2-cobordism can be obtained by composition and disjoint union of the first five generators listed above.*

Proof. The normal form is a recipe for constructing any connected cobordism from the first five generators, without the twist. □

We will need the relations to show explicitly that any connected cobordism can be brought into normal form. We shall therefore postpone this task to Chapter 7

Lemma 5.5. *Every 2-cobordism factors as a permutation cobordism, followed by a disjoint union of connected cobordisms, followed by a permutation cobordism.*

Proof. Let $M : \mathbf{n} \to \mathbf{m}$ be a cobordism. Therefore, M is a 2-manifold with in-boundary $(\partial M)_{\text{in}} = \Sigma_1 \coprod \cdots \coprod \Sigma_n$ and out-boundary $(\partial M)_{\text{out}} = \Sigma'_1 \coprod \cdots \coprod \Sigma'_m$, where all Σ's are all copies of S^1 but can be distinguished with labels, in light of 4.7.

Without loss of generality, let M have exactly two connected components, $M \cong M_0 \coprod M_1$. Then $(\partial M_0)_{\text{in}}$ a subset $\mathbf{p}$ of $(\partial M)_{\text{in}} = \Sigma_1 \coprod \cdots \coprod \Sigma_n$ and $(\partial M_1)_{\text{in}}$ its complement $\mathbf{q}$. It is generally not true that the first p circles belong to M_0 and the last $q = n - p$ circles belong to M_1. But then we can take a diffeomorphism $\mathbf{n} \xrightarrow{\sim} \mathbf{n}$ such that $\mathbf{p}$ comes before (above) $\mathbf{q}$. This diffeomorphism induces a cobordism $S : \mathbf{n} \to \mathbf{n}$, which composes to the cobordism $SM : \mathbf{n} \to \mathbf{m}$ such that $(\partial SM)_{\text{in}}$=$(\partial SM_0)_{\text{in}} \coprod (\partial SM_1)_{\text{in}}$.

Applying now the argument to the out-boundary of SM (which is the out-boundary of M), there exists a permutation cobordism $T : \mathbf{m} \to \mathbf{m}$ such that $SMT : \mathbf{n} \to \mathbf{m}$ is a cobordism which is a disjoint union of connected components as a cobordism. □

Proof. Version 1 of Theorem 5.1 (Normal form) Let M be connected as a manifold. Then it is generated by the first five cobordisms, as has been shown in 5.4. If M has several connected components, it factors into permutation cobordisms and a disjoint union of connected cobordisms. Since the symmetric group is generated by transpositions (see 3.3), the permutation cobordisms can be obtained by composition and disjoint of the twist cobordism and the cylinder/identity cobordism. □

5.2 Morse-theoretic proof

Definition 5.6. A **Morse function** is a smooth map $f : M \to I$, M a manifold, I a real interval, such that all critical points (points where the differential vanishes) are non-degenerate (the determinant of the Hessian matrix does not vanish).

The **index** of a critical value is the number of negative eigenvalues of the Hessian matrix at that point.

For our purposes, let's further assume that $f^{-1}(\partial I) = \partial M$, and that the two points in ∂I are regular values (∂M does not have any critical points).

The next statement follows directly from the Regular Interval Theorem (see talk 3):

Corollary 5.7. *If a cobordism admits a Morse function without critical points then it is equivalent to a permutation cobordism.*

Lemma 5.8. *Let M be a compact, connected, orientable surface with a Morse function $M \to [0, 1]$. If there is a unique critical point x and x has index 1 (x then is a saddle point), then M is diffeomorphic to a disc with two discs missing.*

A disc with two discs missing is diffeo to a sphere with 3 holes and is diffeo to a pair of pants.

Proof. Version 2 of Theorem 5.1 (Morse-theoretic) Let $M : \Sigma_0 \to \Sigma_1$ be a cobordism, $f : M \to [0, 1]$ such that $f^{-1}(0) = \Sigma_0$, $f^{-1}(1) = \Sigma_1$. Since M is compact (in our discussion), f has only finitely many critical values. This allows us to partition $I = \bigcup_{i=0}^{k} [a_i, a_{i+1}]$ such that every subinterval contains at most one critical point.

Consider a subinterval $[a, b] \subset I$, whose preimage $f^{-1}([a, b]) \equiv M_{[a,b]}$ contains at most one critical point x. $M_{[a,b]}$ may consist of several connected components, (at most) one of them contains x, by 5.7 the others are equivalent to permutation cobordisms, which are generated by the twist cobordism and the cylinder (see 3.3).

Thus assume now $M_{[a,b]}$ is connected and has a unique critical point x. If x has index 0 or 2, we have a local minimum or maximum, and $M_{[a,b]}$ is diffeomorphic to a disc. If x has index 1, $M_{[a,b]}$ is diffeomorphic to a pair of pants by 5.8. □

6 Relations for *2Cob*

Recall 3.3, where we first listed the generators graphically for σ_4, and then showed how some compositions of generators are equivalent to others. These equivalences are what we call relations.

Theorem 6.1. *The following relations hold:*

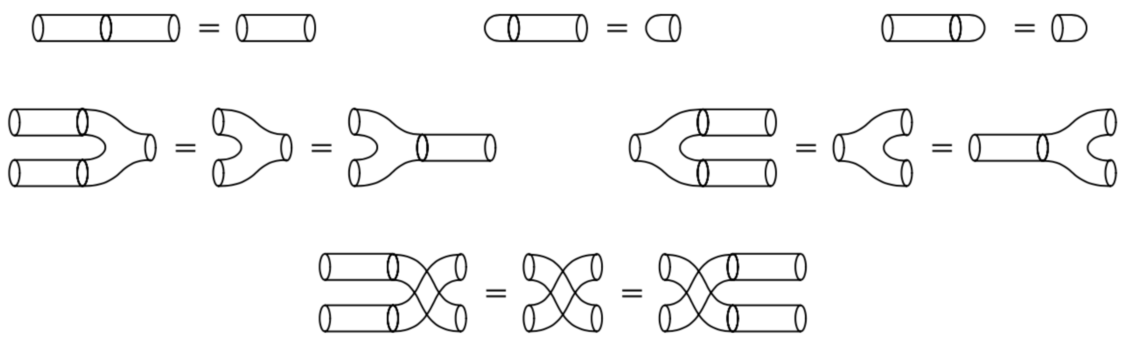


Figure 7: Identity relations.

Figure 8: Disc removal relations.

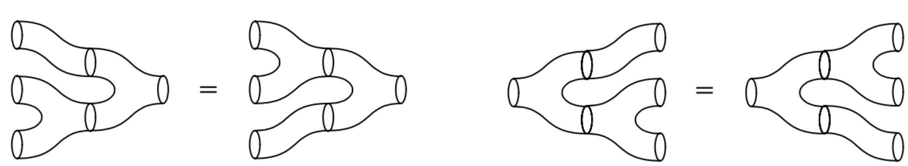


Figure 9: (Co)associativity relations.

Figure 10: (Co)commutativity relations.

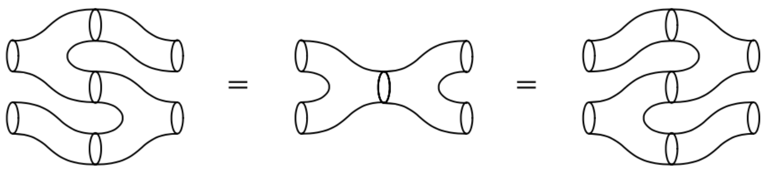


Figure 11: Frobenius relations.

Proof. In each case, the surfaces have the same topological type: they have genus 0 and possess the same number of in- and out-boundaries. Thus by the classification of surfaces, there are diffeomorphic. □

Not satisfied with this argument? Well, you are in luck! There are not one, but two different intuitions to visualise the truth of the various relations. We will not be going through each of them, but we shall showcase the two approaches on one relation each.

Example 6.2. *Note that the cap is diffeomorphic to a disc, the cylinder is diffeomorphic to a disc missing a disc, and the pants are diffeomorphic to a disc missing two discs (with appropriate orientation of their boundaries). The associativity relation is then obtained by making two different decompositions of a disc with three missing discs.*

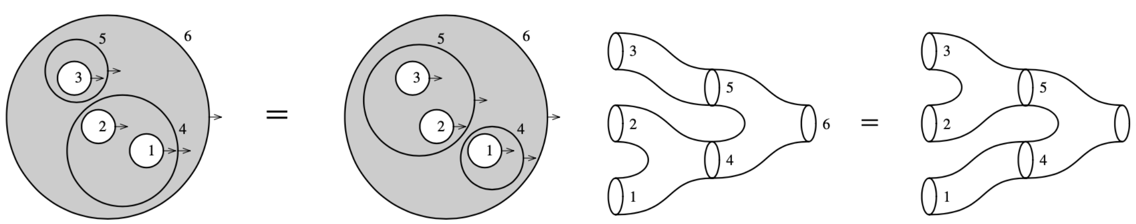


Figure 12: Proof of the associativity relation via nested discs

We can demonstrate the disc removal relations and commutativity in exactly the same way. In the latter case, just note that the pair of pants is diffeomorphic to a disc with two smaller discs within missing, and that we can freely slide them around. To show the right-hand counterparts of those relations, simply reverse the orientation of the boundaries.

Example 6.3. *The demonstration of the Frobenius relation is tricky with discs. Thus note that we have a surface which we can cut up in three different ways.*

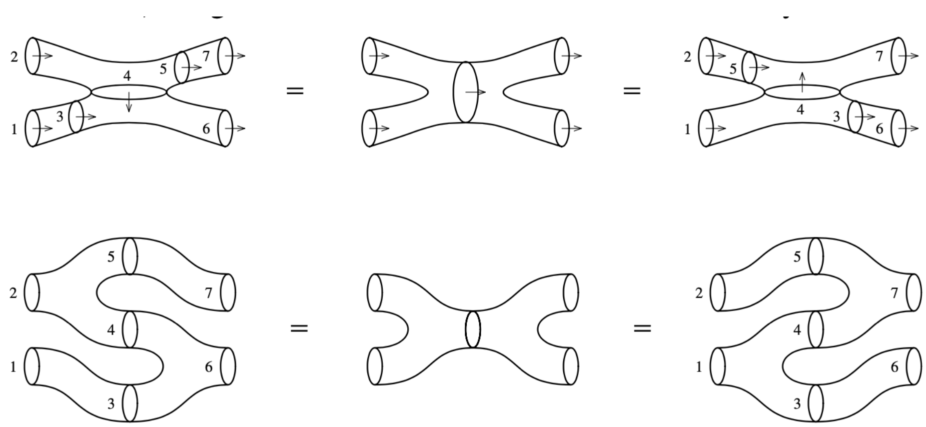


Figure 13: Proof of the Frobenius relation via decomposition

As previously stated, $(\boldsymbol{2Cob}, \coprod, \varnothing, twist)$ is a symmetric monoidal category. In our context, this means that the twist is its own inverse. This combined with the interplay with the other generators leads to additional relations:

Theorem 6.4. *The following relations hold:*

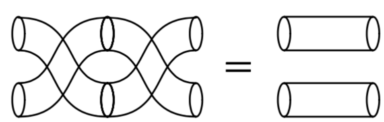


Figure 14: Involutive twist relation.

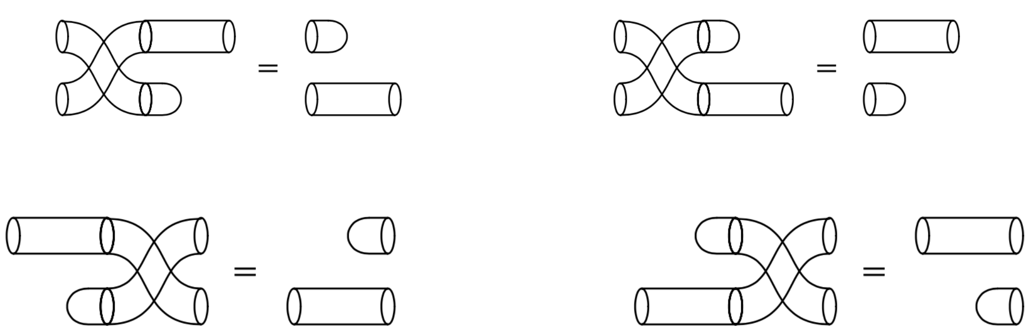


Figure 15: "Moving twist past a cap" relations

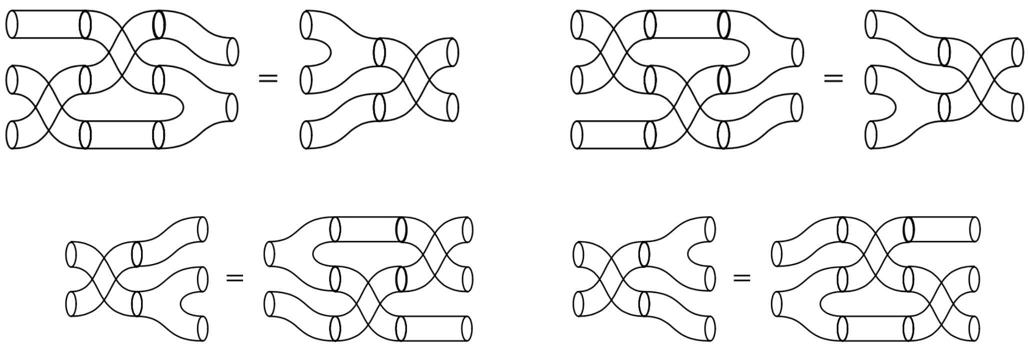


Figure 16: "Moving twist past the multiplication pants" relations

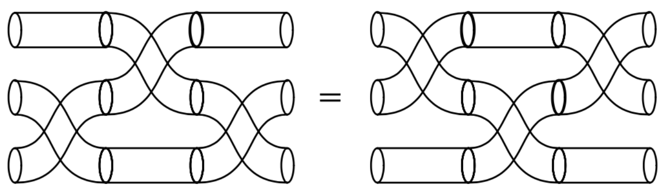


Figure 17: Symmetric group relation

Remark 6.5. *Notice how the relations in Figure 14 and 17 mirror exactly the relations seen in Figure 4.*

7 Sufficiency of the chosen relations

A priori there are infinitely many relations. For example, a four-fold composition of the twist cobordism is equivalent to the identity cylinder as well. But this can already be derived from the fact that the twist is self-inverse, so nothing new under the sun there. The natural question therefore is whether the relations listed in the previous chapter suffice to derive any other equivalences one might think of. Before that, we have a small geometrical consideration to take care of.

Definition 7.1. The **Euler characteristic** of a triangulated surface M is

$$\chi(M) = V - E + F$$

where V, E, F are the number of vertices, edges, and faces of the triangulation, respectively.

Lemma 7.2. *Let M be a connected, compact surface with genus g and k boundaries. Then its Euler characteristic is*

$$\chi(M) = 2 - 2g - k$$

Proof. We start with the following closed surfaces: a sphere S^2 with $\chi(S^2) = 2$, and a torus with $\chi(T^2) = 0$. Both Euler characteristics can be computed easily by how we calculated the Euler characteristic of CW-complexes (see second introductory lecture),

$$\chi(M) = \sum_{k=0}^{n} (-1)^k m_k \, .$$

Since for connected sums, the following relations holds,

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^1) = \chi(M_1) + \chi(M_2) - 2 \, ,$$

we get for the connected sum of n tori (i.e. an n-holed hollow doughnut)

$$\chi(\#^n T^2) = n\chi(T^2) - 2(n-1) = -2(n-1), \quad n \in \mathbb{N}_0 \, .$$

Therefore, starting with a sphere with Euler characteristic 2, for every hole (genus), we reduce by 2. The Euler characteristic of closed surfaces with genus g is thus $2 - 2g$. Now, for every boundary component, we remove one disc. By additivity,

$$\chi(M_1 \cup M_2) = \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2) \, ,$$

we have

$$\chi(M \backslash D^2) = \chi(M) - \chi(D^2) = \chi(M) - 1 \, ,$$

which we can repeat for k boundary components, and we conclude. □

Proposition 7.3. *The relations listed in 6.1 are sufficient for connected cobordisms.*

Proof. Let M be a connected surface with n in-boundaries, m out-boundaries, and genus g. Furthermore let a be the number of left pants, b be the number of right pants, p be the number of left discs, and q be the number of right discs.

By 7.2 and the additivity respectively, we state that the Euler characteristic suffices

$$\chi(M) = 2 - 2g - n - m = p + q - a - b \, .$$

On the other hand we can count the contributions to the number of in- and out-boundaries,

$$\begin{aligned} n &= b + 2a - p \, , \\ m &= a + 2b - q \, , \end{aligned}$$

and we infer

$$a + q + m = b + p + n \, .$$

Combining the two equations yields

$$\begin{aligned} a &= n - 1 + g + p \, , \\ b &= m - 1 + g + q \, . \end{aligned}$$

The plan now is thus to take $n-1$ left pants and move them to the left until they become before any right pants in order to form the in-part of the normal form. If we meet a left cap, then by disc removal (Figure 8) it becomes a cylinder. This happens p times, so we have enough left pants to spend. If we meet a pair of right pants, they either connect by one trouser leg or both. In the first case, the Frobenius relation (Figure 11) applies, and we can move past them. In the second case, they form a so-called handle or genus hole. This will happen g times. To move left pants past a handle, we first use associativity (Figure 9) and then Frobenius. Filling up with cylinders as appropriate, we end up with an in-part consisting of $n-1$ left pants, and a total of p in-boundaries according to normal form.

Repeating equivalently with right pants yields the out-part consisting of $m-1$ right pants, and a total of q out-boundaries according to normal form. In the middle-part, we eventually have a chain of g handles, with one in- and one out-boundary.

To eliminate twist maps in connected surfaces, consider an arbitrary twist. If the both left-hand parts are connected, they form a surface with strictly less twist maps, so they can be brought in normal form inductively using the relations. Since only the out-part of the left-hand side touches the twist, we can use one of the right pants to eliminate the twist via cocommutativity (Figure 10. If both right-hand parts are connected the same argument applies with one pair of left pants and commutativity. If the top or bottom ends of the twist are connected, the situation will be close to 7.4 (Figure 18). One of the scenarios has to apply since otherwise the surface would not be connected, and we conclude. □

Example 7.4. *Here, we eliminate the twist by applying cocommutativity, then "moving twist past the multiplication pants", then Frobenius, and finally cocommutativity again.*

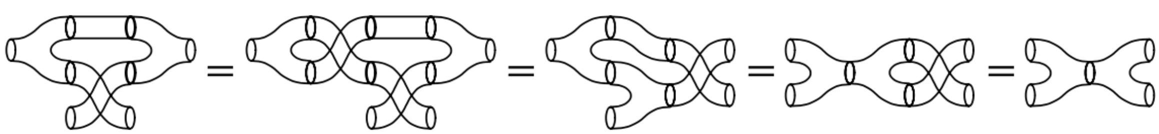


Figure 18: Example of elimination of twist map.

All we have to do now is to consider the case of non-connected surfaces and prove sufficiency there as well.

Definition 7.5. The **normal form** of any oriented (& compact) surface with n in-boundaries, m out-boundaries, and genus g is a cobordism which factorises into a permutation cobordism $\mathbf{n} \to \mathbf{n}$ on the left, a disjoint union of normal forms of connected cobordisms in normal form 5.2, and a permutation cobordism $\mathbf{n} \to \mathbf{n}$ to the right.

Theorem 7.6. *The relations listed in 6.1 and 6.4 are sufficient for all cobordisms.*

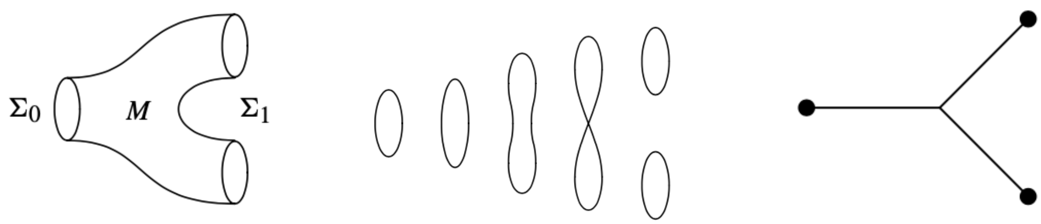
Proof. As in the proof of 5.5, we find permutation cobordisms $S : \mathbf{n} \to \mathbf{n}$ and $T : \mathbf{m} \to \mathbf{m}$ such that $SMT : \mathbf{n} \to \mathbf{m}$ is equivalent to a cobordism which is a disjoint union of connect cobordisms as a cobordism. Any connected middle piece can be brought into normal form, thus we gathered the normal form for general surfaces, and we are finished. □

8 Digestif

So there you have it. In summary, we have:

1. Constructed a skeleton of ***2Cob*** and argued why this suffices
2. Found generators and relations for this skeleton
3. Demonstrated the sufficiency of said generators and relations.

With this powerful tool in hand, we can now embark on a journey to discuss more advanced topics in TQFT. As in the introduction stated, this is a toy model and mathematical framework to discuss quote-on-quote ”real” theories such as Quantum Field Theory in physics, which is a very general theory of how particles move, propagate, and behave. A left pair of pants cobordism for example can model how over time, a particle decays into two particles, see Figure 19.



(a) From a cobordism... (b) ...over propagation in time... (c) ...to a Feynman diagram

Figure 19: Illustration of the connection between cobordisms and physical processes.

One example of such a decay process is the Higgs boson discovered in 2012, which amongst many other decay modes, has been observed to decay into a pair of W bosons,

$$H \to W^+ + W^- \,.$$

Another more well-known example is the radioactive β^- decay, where a neutron decays into a proton, an electron, and an electron antineutrino,

$$n \to p + e^- + \bar{\nu}_e \,,$$

which we can describe as a cobordism with one in-boundary and three out-boundaries.

References

[1] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories*. Cambridge University Press, 2003. ISBN: 978-0-511-07727-2.