Physical Derivation of Index Theorem
MAT782 Spin Geometry

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Abstract

We present a quick physical derivation of the Atiyah-Singer index theorem using $N = 1$ supersymmetric quantum mechanics.

Contents

1 Lightning fast introduction to SUSY QM

So far, the theory of supersymmetry has failed to be validated in a real experimental setting. However, the lack of practicality of this framework is more than compensated by the rich theoretic and mathematical insights one can gain while studying this theory. A prominent example is one of many ways one may derive or prove the Atiyah-Singer index theorem, which we will discuss here.

1.1 The spectrum of supercharges

Definition 1.1. By slight abuse of convention, we call a Hamiltonian H **supersymmetric** if there exists an operator Q called **supercharge** such that

$$H = \frac{1}{2}\{Q, Q^\dagger\} \quad \text{and} \quad Q^2 = 0\,.$$

Proposition 1.2. *The energy eigenvalues of a supersymmetric Hamiltonian are non-negative. Further, the energy is zero, called a ground state ψ, if and only if $|\psi\rangle$ is annihilated by both Q and $Q^\dagger$.*

Proof. Let ψ be a state in the Hilbert space. Then

$$E = \langle\psi|H|\psi\rangle = \frac{1}{2}\langle\psi|Q^\dagger Q + QQ^\dagger|\psi\rangle = |Q|\psi\rangle|^2 + |Q^\dagger|\psi\rangle|^2 \geq 0\,,$$

and $E = 0 \iff Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$. □

Lemma 1.3. *Consider the set of states with some fixed energy E, $H|\psi\rangle = E|\psi\rangle$. Then $[H, Q] = [H, Q^\dagger] = 0$.*

Proof. Using that Q and $Q^\dagger$ square to zero, we have

$$2H = QQ^\dagger + Q^\dagger Q \implies 2HQ = QQ^\dagger Q = Q\big(\{Q, Q^\dagger\} - QQ^\dagger\big) = 2QH\,.$$

□

The whole idea of supersymmetry is that there are two types of particles, bosons and fermions, and that they behave fundamentally different (indeed they do, they follow different statistics, and the Pauli exclusion principle holds only to fermions).

Remark 1.4. *For fixed $E > 0$, we introduce the normalised notion of the supercharge $c = Q/\sqrt{2E}$, with $\{c, c^\dagger\} = 1$ (and of course $c^2 = (c^\dagger)^2 = 0$). This is the algebra formed by fermionic creation and annihilation operators, where we denote the basis of the two-dimensional irreducible representation by the states $|0\rangle$ and $|1\rangle$ (they span the representation). We have $c|0\rangle = 0$ and $c^\dagger|0\rangle = |1\rangle$.*

Since H and Q commute, H and c commute. Hence have $|0\rangle$ and $|1\rangle$ the same energy eigenvalue $E > 0$, but as we will see shortly, with differing fermionic number.

Corollary 1.5. *All excited ($E > 0$) states come in pairs. This holds generally not for ground ($E = 0$) states $|\Omega\rangle$, since by Prodopsition 1.2 necessarily $Q|\Omega\rangle = Q^\dagger|\Omega\rangle = 0$ and hence $c|\Omega\rangle = c^\dagger|\Omega\rangle = 0$, so we cannot automatically create new states with energy equal zero.*

1.2 Bosons and fermions

Definition 1.6. For $E > 0$, we define the **fermion number operator** as $F = c^\dagger c$. The operator $(-1)^F$ is called **fermion parity**.

Lemma 1.7. *F obeys $[F, Q] = -Q$, $[F, Q^\dagger] = Q^\dagger$, $[F, H] = 0$, $F|0\rangle = 0$ and $F|1\rangle = |1\rangle$.*

Corollary 1.8. *The Hilbert space decomposes into bosonic states with $F = 0$ and fermionic states with $F = 1$, $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$.*

Remark 1.9. *This is a $\mathbb{Z}_2$ grading of the Hilbert space. The $E > 0$ states have one state in $\mathcal{H}_B$ and one state in $\mathcal{H}_F$. It is not yet clear which of the spaces we should assign the $E = 0$ states to.*

Corollary 1.10. *The unitary operator $(-1)^F$ is involute (it squares to the identity), takes eigenvalue $+1$ on states in $\mathcal{H}_B$ and eigenvalue -1 on states in $\mathcal{H}_F$. Equivalently, it commutes with bosonic operators and anticommutes with fermionic operators.*

Remark 1.11. *A physical potential can be modelled or at least approximated by a real polynomial h. Varying the parameters of h continuously results in a change of the energy spectrum of the Hamiltonian, with the exception of the ground states. They are in a sense very stable.*

2 The Witten index

2.1 Mathematical interlude

Remark 2.1. *Mathematically, the difference between the physical behaviour between bosons and fermions can be made clear as well. The classic example of a $\mathbb{Z}_2$ graded vector space is $\mathbb{R}^{p|q} \cong \mathbb{R}^p \oplus \mathbb{R}^q$, where declare elements of $\mathbb{R}^p$ to have parity 0 and element of $\mathbb{R}^q$ to have parity 1 (hence the parity operation is given by F). To turn $\mathbb{R}^{p|q}$ into a superalgebra, we require*

$$[x^i, x^j] = 0\,, \quad [x^i, \psi^a] = 0\,, \quad \text{and} \quad \{\psi^a, \psi^b\} = 0\,,$$

where we call $x^i \in \mathbb{R}^{p|0}$ bosonic and $\psi^a \in \mathbb{R}^{0|q}$ fermionic. Bosons are standard, real variables, while fermions are so-called (complex) Grassmann variables. If we impose $\bar{\psi} = \psi$, we call them Majorana modes. For $\mathbb{R}^{p|q}$, $\partial/\partial x^i$ (the usual derivative on $\mathbb{R}^p$) are even derivatives, while $\partial/\partial\psi^a$ are odd derivatives, with

$$\frac{\partial x^j}{\partial x^i} = \delta_i^j \quad \frac{\partial \psi^b}{\partial \psi^a} = \delta_a^b \quad \frac{\partial x^j}{\partial \psi^a} = \frac{\partial \psi^b}{\partial x^i} = 0$$

and

$$\frac{\partial}{\partial \psi^a}(\psi^b \psi^c) = \delta_a^b \psi^c - \psi^b \delta_a^c\,.$$

Remark 2.2. *One of the most significant differences between bosons and fermions is the following. Let D be a generic differential operator. Then the Gaussian integral for bosonic variables is $\int e^{-x^i D x^j} \sim \sqrt{1/\det(D)}$, and for fermionic variables is $\int e^{-\psi^a D \psi^b} \sim \pm\sqrt{\det(D)}$.*

2.2 The many forms of the Witten index

Definition 2.3. The **Witten index** of a theory is the difference between the number of bosonic and fermionic ground states,

$$\mathcal{I} \equiv \mathrm{Tr}\big((-1)^F e^{-\beta H}\big)$$

This formula needs quite a bit of unpacking. First of all, we are taking this trace over the whole Hilbert space. Notice that the fermion parity causes the sign of the values to alternate, depending on whether the state is bosonic or fermionic.

Secondly, the exponential is very close to the time evolution operator e^{-itH}, which as the name suggests, describes how a wavefunction evolves over time. We will often use the procedure of turning the time imaginary, $t \to -i\tau$, which is called a Wick rotation, with τ the Euclidean time. Hence the operator describes the evolution to the Euclidean time $\tau = \beta$[1]. There is a very surprising result to consider.

Lemma 2.4. *The index is independent of* β*,*

$$\frac{d\mathcal{I}}{d\beta} = 0\,.$$

This is clear for the excited states, as the factor of $(-1)^F$ ensures that their contributions cancel out.

The ground states generally do not cancel, but contrary to the excited states, are not sensitive to the value of β (compare this to Remark 1.11).

Lemma 2.5. *Formally, we have the isomorphism*

$$\mathcal{H}_B\Big|_{E>0} \cong \mathcal{H}_F\Big|_{E>0}\,.$$

Corollary 2.6. *The Witten index counts the difference in the number of ground states in both subspaces [1],*

$$\mathcal{I} = \mathrm{dim}\mathcal{H}_{0,B} - \mathrm{dim}\mathcal{H}_{0,F} = n_B^{E=0} - n_F^{E=0}\,.$$

Proposition 2.7. *Let* $\hat{A}$ *be any operator,* $|\eta\rangle = e^{\hat{\bar{\psi}}\eta}|0\rangle$ *and* $\langle\bar{\eta}| = \langle 0|e^{\hat{\psi}\bar{\eta}}|$ *be the fermionic coherent states.*

The following identities hold:

$$1_{\mathcal{H}} = \int e^{-\bar{\eta}\eta}|\eta\rangle\langle\bar{\eta}|d^2\eta \quad \text{and} \quad \langle\bar{\eta}|\eta\rangle = e^{\bar{\eta}\eta}\,,$$

$$\mathrm{Tr}\big(A\big)_{\mathcal{H}} = \int e^{-\bar{\eta}\eta}\langle-\bar{\eta}|A|\eta\rangle d^2\eta\,,$$

$$\mathrm{STr}(A)_{\mathcal{H}} = \mathrm{Tr}\big((-1)^F A\big) = \int e^{-\bar{\eta}\eta}\langle\bar{\eta}|A|\eta\rangle d^2\eta$$

[1]Compare this to the "inverse temperature" in statistical mechanics.

Proposition 2.8. *Let H a supersymmetric Hamiltonian, S its associated action depending on a collection of bosonic and fermionic variables ϕ. Then, $\forall \beta > 0$,*

$$\mathrm{Tr}\big((-1)^F e^{-\beta H}\big) = \int_P e^{-S[\phi]} \mathcal{D}\phi \,,$$

where P the space of all paths over periodic Euclidean time [2].

Proof. Let χ describe a coherent state. The amplitude (heat kernel) for them is defined by $\langle \bar{\chi}' | e^{-\beta H} | \chi \rangle$. Let the Euclidean time $\beta = N\Delta\tau$. By non-trivially commuting if necessary, we get

$$\begin{aligned}
\langle \bar{\chi}' | e^{-\beta H(\hat{\bar{\psi}}, \hat{\psi})} | \chi \rangle &= \langle \bar{\chi}' | e^{-\Delta\tau H} \dots e^{-\Delta\tau H} | \chi \rangle = \langle \bar{\chi}' | e^{-\Delta\tau H} 1_{\mathcal{H}} e^{-\Delta\tau H} \dots 1_{\mathcal{H}} e^{-\Delta\tau H} | \chi \rangle \\
&= \int \langle \bar{\chi}' | e^{-\Delta\tau H} | \eta_{N-1} \rangle \langle \bar{\eta}_{N-1} | e^{-\Delta\tau H} | \eta_{N-2} \rangle \dots \langle \bar{\eta}_1 | e^{-\Delta\tau H} | \chi \rangle \prod_{k=1}^{N-1} e^{-\bar{\eta}_k \eta_k} d^2\eta_k \\
&= \lim_{N\to\infty} \int \exp\Big(\sum_{k=1}^{N} \bar{\eta}_k \eta_{k-1} - \Delta\tau H(\bar{\eta}_k, \eta_{k-1}) \Big) \prod_{k=1}^{N-1} e^{-\bar{\eta}_k \eta_k} d^2\eta_k \\
&= \lim_{N\to\infty} \int \exp\Big(- \sum_{k=1}^{N} \Big[\bar{\eta}_k \frac{\eta_k - \eta_{k-1}}{\Delta\tau} + H(\bar{\eta}_k, \eta_{k-1}) \Big] \Delta\tau \Big) e^{\bar{\eta}_N \eta_N} \prod_{k=1}^{N-1} d^2\eta_k \\
&= \int e^{-S_E[\psi, \bar{\psi}]} e^{\bar{\psi}(\beta)\psi(\beta)} \mathcal{D}\psi \mathcal{D}\bar{\psi}
\end{aligned}$$

using in the third line that $|\eta_k\rangle$ and $\langle \bar{\eta}_k|$ are eigenstates of $\hat{\psi}$ and $\hat{\bar{\psi}}$, respectively, thus for infinitesimal $\Delta\tau$

$$\langle \bar{\eta}_{k+1} | e^{-\Delta\tau H(\hat{\bar{\psi}}, \hat{\psi})} | \eta_k \rangle = e^{-\Delta\tau H(\bar{\eta}_{k+}, \eta_k)} \langle \bar{\eta}_{k+1} | \eta_k \rangle = e^{-\Delta\tau H(\bar{\eta}_{k+}, \eta_k)} e^{\bar{\eta}_{k+1}\eta_k} \,,$$

and in the fifth line that the inside of the square bracket in the fourth line corresponds to a discretisation of the Euclidean action,

$$S_E[\eta, \bar{\eta}] = \int_0^\beta \Big[\bar{\eta}\dot{\eta} + H(\bar{\eta}, \eta) \Big] d\tau \,.$$

In the final integral, we integrate over all $\psi(\tau)$ s.t. $\psi(0) = \chi$ and $\psi(\beta) = \chi'$.

So, in total, we have

$$\begin{aligned}
\mathrm{STr}(e^{-\beta H})_{\mathcal{H}} = \mathrm{Tr}\big((-1)^F e^{-\beta H}\big) &= \int e^{-\bar{\eta}\eta} \langle \bar{\eta} | e^{-\beta H} | \eta \rangle d^2\eta \\
&= \int e^{-S_E[\psi, \bar{\psi}]} \mathcal{D}\psi \mathcal{D}\bar{\psi}
\end{aligned}$$

where we formally integrate over all $\psi(\tau)$ s.t. $\psi(0) = \chi$ and $\psi(\beta) = \chi'$.

A similar result for bosonic variables can be shown in the same vein, so combining them yields

$$\mathrm{Tr}\big((-1)^F e^{-\beta H}\big) = \int_P e^{-S_E[x,\psi,\bar{\psi}]}\mathcal{D}x\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

where since we are taking the trace, we really should only be considering $\psi(\tau)$ with $\psi(0) = \psi(\beta)$ (similar for x). This is equivalent of taking the integral is over fields living on a circle of circumference β, i.e. the fermions and bosons that are periodic around the S^1 worldline. This is called the path integral representation of the Witten index. □

3 Non-linear σ-models

To set the stage, let (N, g) be a (compact) Riemannian manifold with dim $N = n$. We denote by x^a the local coordinates for N, with $a = 1, \ldots, n$.

Consider the nonlinear σ-model[2], which classically describes a field taking values on a non-linear manifold. The reason for this is that the only spinors that solve the free Dirac equation $\partial\!\!\!/\chi = 0$ in flat space are constant, hence unphysical. The supersymmetric extension of this action involves ψ^a be n different complex fermions.

We may assume $\bar{\psi} = \psi$, which corresponds to $N = 1$ ($\mathcal{N} = 1/2$) supersymmetry, a system where we have a single real conserved supercharge Q.

Proposition 3.1. *Let $\bar{\psi} = \psi$. Then the (Euclidean) action of the non-linear σ-model simplifies to*

$$S_E[x, \psi] = \int \frac{1}{2} g_{ab}\dot{x}^a\dot{x}^b + \frac{1}{2} g_{ab}\psi^a\nabla_\tau\psi^b d\tau\,.$$

with

$$\nabla_\tau\psi^a = \frac{d}{d\tau}\psi^a + \Gamma^a_{bc}\frac{dx^b}{d\tau}\psi^c$$

the covariant derivative. The first term describes the bosonic, the second term the fermionic kinetic energy, R the Riemann curvature of N.

Theorem 3.2. *The conserved supercharge is the Dirac operator,*

$$Q = \psi^a p_a = i D\!\!\!\!/$$

and the Hamiltonian is $H = Q^2 = -D\!\!\!\!/^2$.

Remark 3.3. *Formally, the Hilbert space of the bosonic field is the space of square-integrable functions on N with respect to the measure $\sqrt{g}d^n x$, $\mathcal{H}_B = L^2(N, \sqrt{g}d^n x)$. The natural quantization of the fermionic fields ψ is the space of Dirac spinors $\mathcal{H}_\psi = S$, with the fermions acting as Dirac γ-matrices (since they both follow the Clifford algebra). Thus in total, the Hilbert space is*

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F \cong L^2(S(N), \sqrt{g}d^n x)\,,$$

the space of square-integrable sections of the spin bundle on N.

[2] σ denotes a generic spinless meson

Lemma 3.4. *For a valid quantum theory, we require the number of Majorana modes to be even [3].*

Proof. From Remark 1.4 we know that two Majorana modes ψ^1, ψ^2 act on a 2-dimensional Hilbert space (since $c^\dagger = (\psi^1 - i\psi^2)/\sqrt{2}$ acts as raising operator), with $\mathcal{H}_{1,2} = \mathcal{H}_1 \otimes \mathcal{H}_2$. By the Hilbert space tensor product rule, $2 = \dim\mathcal{H}_{1,2} = \dim\mathcal{H}_1 \cdot \dim\mathcal{H}_2$. So in a sense, a single Majorana mode acts on a "$\sqrt{2}$-dimensional Hilbert space". In particular in our framework, we cannot allow an odd number of Majorana modes. □

Hence for our purposes, from now on we assume $\dim N = n = 2m$ to be even. In particular, we require the number of Majorana modes to be even. This means that we can split the space of the Dirac spinors into $+1$ and -1 eigenspace of the chirality operator. This can be written as $\gamma^{n+1} \equiv i^{n/2}\gamma^1 \ldots \gamma^n$ and further be identified as $(-1)^F$. In summary, we decompose the space of Dirac spinors as $S = S^+ \oplus S^-$.

Remark 3.5. *We have the vielbein e^i_a up to $SO(n)$ as $g_{ab} = \delta_{ij} e^i_a e^j_b$ which defines an orthonormal frame at each $p \in N$, with inverse e^a_i. Introduce spin connection 1-form ω^i_j by requiring*

$$\nabla_a e^i_b = \partial_a e^i_b - \Gamma^c_{ab} e^i_c + \omega^i_{aj} e^j_b = 0$$

thus

$$w^i_{aj} = \frac{1}{2} e^b_j (\partial_a e^i_b - \partial_b e^i_a) - \frac{1}{2} e^{bi} (\partial_a e_{bj} - \partial_b e_{aj}) - \frac{1}{2} e^{bi} e^{cj} (\partial_b e_{kc} - \partial_c e_{kb}) e^k_a \,.$$

Note that as usual, the raising and lowering of indices is done by contracting with the metric. The metric on the tanget space is δ_{ij}. We define the curved space Dirac operator

$$\not{D}\psi = \gamma^a(\partial_a \psi + \omega^{jk}_a \Sigma_{jk} \psi)$$

to parallel transport spinors on N. Since $\gamma^{n+1} = \operatorname{diag}(\mathbb{I}, -\mathbb{I})$ and $\{\not{\nabla}, \gamma^{n+1}\} = 0$ the Dirac operator maps even spinors to odd spinors and vice versa, we can write

$$\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix}$$

with respect to the decomposition $S = S^+ \oplus S^-$, with $\not{D}^\pm : S^\pm \to S^\mp$, $\not{D}^- = (\not{D}^+)^\dagger$, and $(\not{D}^+)^2 = (\not{D}^-)^2 = 0$.

Definition 3.6. We define the **index** of the Dirac operator as

$$\operatorname{ind}\not{D}^+ = \dim\ker(\not{D}^+) - \dim\ker(\not{D}^-) \,.$$

and we call $\not{D}^+$ the **Atiyah-Singer operator**.

The crucial observation is now the following: The kernel of $\not{D}^+$ ($\not{D}^-$) is the space of all bosons (fermions) that are mapped to zero, i.e. fulfil the Dirac equation $\not{D}\chi = 0$. But those solutions are exactly the ground states of the quantum theory!

Theorem 3.7. *The index of the Dirac operator coincides with the Witten index,*

$$\operatorname{ind}\not{D}^+ = \dim\ker(\not{D}^+) - \dim\ker(\not{D}^-) = n_B^{E=0} - n_F^{E=0} = \operatorname{Tr}\big((-1)^F e^{-\beta H}\big) = \int_P e^{-S_E[x,\psi]} \mathcal{D}x \mathcal{D}\psi = \mathcal{I} \,.$$

4 Derivation of the index theorem

Theorem 4.1 (Atiyah-Singer index theorem)**.** *Let N be an even-dimensional spin manifold. Then*

$$\text{ind}\not{D}_N^+ = \int_N \hat{A}\Big(\frac{iR_N}{2\pi}\Big)\,.$$

Proof. We want to explicitely compute the path integral representation of the index,

$$\text{ind}\not{D}^+ = \int_P e^{-S_E[x,\psi]}\mathcal{D}x\mathcal{D}\psi\,.$$

As usual, x and ψ periodic implies that the path integral over fields on S^1 is independent of the circumference β.

Thus we can take the limit $\beta \to 0$, wherein constant field configurations (x_0, ψ_0) dominate. This can be seen by rescaling $\tau \to \tau/\beta$ and $\psi \to \beta^{-1/4}\psi$. Then for $\beta \to 0$, the argument $-S_E$ in the exponential does not go to negative infinity if and only if $\dot{x} = \dot{\psi} = 0$. Hence we expand as $x^a(\tau) = x_0^a + \delta x^a(\tau)$, $\psi^a(\tau) = \psi_0^a + \delta\psi^a(\tau)$, with $\oint \delta x^a(\tau)d\tau = \oint \delta\psi^a(\tau)d\tau = 0$. Using Riemann normal coordinate expansion for metric and connection,

$$\begin{aligned}
g_{ab}(x) &= \delta_{ab} - \frac{1}{3}R_{acbd}(x_0)\delta x^c\delta x^d + \mathcal{O}(\delta x^3)\\
\Gamma^a_{bc}(x) &= -\frac{1}{3}(R^a_{bcd}(x_0) + R^a_{cbd}(x_0))\delta x^d + \mathcal{O}(\delta x^2)\,,
\end{aligned}$$

we get

$$S_E[x_0,\psi_0,\delta x,\delta\psi] = \oint -\frac{1}{2}\delta x_a\frac{d^2}{d\tau^2}\delta x^a + \frac{1}{2}\delta\psi_a\frac{d}{d\tau}\delta\psi^a - \frac{1}{4}R_{abcd}\psi_0^a\psi_0^b\delta x^c\frac{d\delta x^d}{d\tau}d\tau\,.$$

Integrating over the fluctuations, with $\mathcal{R}^a_b = R^a_{bcd}(x_0)\psi_0^c\psi_0^d$, gives

$$Z \equiv \int e^{-S_E[x_0,\psi_0,\delta x,\delta\psi]}\mathcal{D}\delta x\mathcal{D}\delta\psi = \frac{\sqrt{\det(\delta^a_b\partial_\tau)}}{\sqrt{\det(-\delta^a_b\partial^2_\tau + \mathcal{R}^a_b\partial_\tau)}} = \frac{1}{\sqrt{\det(-\delta^a_b\partial_\tau + \mathcal{R}^a_b)}}$$

This integration is not really obvious, but note that we essentially compute the Gaussian integral of differential operators (compare Remark 2.2).

We can decompose TN_{x_0} into $n/2$ two-dimensional spaces that are invariant under the action of $\mathcal{R}^a_b$ with restrictions to the k-th subspace $\mathbb{W}|_k$,

$$\mathcal{R}^a_b = \begin{pmatrix} \mathbb{W}|_1 & & \\ & \ddots & \\ & & \mathbb{W}|_{n/2}\end{pmatrix} \quad \text{with} \quad \mathbb{W}|_k = \begin{pmatrix} 0 & \omega_k \\ -\omega_k & 0\end{pmatrix}$$

for some ω_k, with eigenvalues $\pm i\omega_k$. We diagonalise the ∂_τ term by working in a Fourier basis around S^1. Since the Witten index is independent of $\beta > 0$, take $\beta = 1$. Then $\delta x^k(\tau) \sim e^{2\pi i p\tau}$

with $p \in \mathbb{Z} \setminus 0$. Hence the eigenvalues for any $\mathbb{W}$ are $i(2\pi p \pm \omega)$, and

$$\sqrt{\det(-\partial_\tau + \mathbb{W})} = \prod_{p \neq 0} \sqrt{2\pi i p + i\omega}\sqrt{2\pi i p - i\omega} = \underbrace{\prod_{p=1}^{\infty} (2\pi i p)^2}_{=-i} \prod_{p=1}^{\infty} \left[1 + \left(\frac{i\omega}{2\pi p}\right)^2\right] = -i \frac{\sinh(i\omega/2)}{i\omega/2} ,$$

computing the first product via ζ-function regularisation. Combining all $n/2$ subspaces in the reciprocal, we have

$$Z = (-i)^{n/2} \prod_{k=1}^{n/2} \frac{i\omega_k/2}{\sinh(i\omega_k/2)} ,$$

and thus in total,

$$\mathrm{ind} \not{D}^+ = \int_P Z \mathcal{D}x_0 \mathcal{D}\psi_0 = (-i)^{n/2} \int \prod_{i=1}^{n} \frac{dx_0^i d\psi_0^i}{\sqrt{2\pi}} \prod_{k=1}^{n/2} \frac{i\omega_k/2}{\sinh(i\omega_k/2)}$$

since the constant fields are just regular variables. This is now a very familiar expression from the lecture [4], and we write

$$\mathrm{ind} \not{D}^+ = \frac{1}{(2\pi)^{n/2}} \int_N \sqrt{\det\left(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}\right)} = \int_N \hat{A}\left(\frac{iR_N}{2\pi}\right) .$$

□

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